An easy proof of Gowers' FIN_k theorem

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ABSTRACT

A new proof of Pigeonhole Principle of Gowers is found. The proof does not use of the concept of ultrafilter.

The purpose of the paper is to give an elementary proof of the theorem of Gowers ([4]) being a generalization of Hindman Theorem ([5]). The presented proof is purely combinatorial and (on the contrary to original Gowers' proof) do not use the theory of ultrafilters as well as the full strength of the Axiom of Choice (AC in short), no uncountable versions of AC are used. Our method is in the spirit of Baumgartner's proof of Hindman Theorem ([2]). In fact our proof stands in the same relation to Baumgartner's as the ultrafilter proof of Hindman Theorem to the proof of Gowers' theorem; we refer the reader to [3] for a comparison. In particular the main ingredient, **Lemma 6**, replaces Gowers' Lemma 3 from [4] (known also as the lemma on stabilization of continuous endomorphisms, cf. [1]).

The demand for a combinatorial proof was raised for example in [10]: *in hindsight one can see that Baumgartner's proof of Hindman's theorem also uses a combinatorial forcing and this suggests the natural question of whether there is an analogous proof that would establish Gowers's pigeon hole principle for* FIN_k*, originally proved using the methods of topological dynamics.* While the presented proof uses Baumgartner's ideas it can hardly be regarded as analogous. It shares more similarities with recently established in [8] and [11] proofs of **Finite** $\mathbf{FIN}_k$ **Theorem**. We stress however that methods from the above papers do not allow to prove $\mathbf{FIN}_k$ **Theorem**.

The paper is organized in two sections. The first one consists of notations and basic facts. In the second section we present a brief history of $GMTT$ theorems and finally we prove Gowers' **FIN**$_k$ **Theorem** without referring to any uncountable versions of AC or its consequences such as the existence of ultrafilters.

1. Preliminaries.

Throughout the paper we use letters i, j, k, l, m, n for nonnegative natural numbers and by ω we denote their universe. Let also $\mathbb{N} := \omega \setminus \{0\}$. We prefer to treat numbers as ordinals; thus for example $n = \{0, 1, \ldots, n-1\}$ and $i < 2$ means $i \in \{0, 1\}$. Then we can define for $k \in \mathbb{N}$

$$Comb_k := k^2 \backslash (k \backslash 1)^2 = \{(i, j) \in k^2 : 0 \in \{i, j\}\} = \{(i, j) \in k^2 : \min\{i, j\} = 0\}.$$

Gowers' Pigeonhole Principle also known as **FIN**$_k$ **Theorem** is a Ramsey-type theorem about particular families of functions (sequences) and that is why we need specific notations; some of them we borrow from [10]. We put $\mathrm{supp}(f) = \{n < \omega : f(n) \neq 0\}$ for the support of a function $f : \omega \to k + 1$, $k < \omega$, while by $\mathrm{rng}(f)$ we we denote its range $f[\omega]$. Define the main object in the paper by

$$\mathrm{FIN}_k = \{f \in^\omega (k + 1) : |\mathrm{supp}(f)| < \omega \ \& \ k \in \mathrm{rng}(f)\}.$$

Thus FIN_0 is a singleton of the null function $\mathrm{FIN}_0 = \{\{(i, 0) : i < \omega\}\} = \{O\}$.

We equip the collections FIN_k, $k \in \mathbb{N}$, with an ordering defined as

$$p < q \quad \text{if} \quad \max \mathrm{supp}(p) < \min \mathrm{supp}(q), \quad p, q \in \mathrm{FIN}_k.$$

Moreover, for functions comparable under this ordering we define their sum as pointwise sum of functions. It makes the structure $(\mathrm{FIN}_k, +)$ a partial semigroup (cf. [10]). Regarding sums we shall use the following conventions. If we write $p + q$ we always implicite assume that $p \cdot q = O$ (pointwise multiplication), i.e. $\mathrm{supp}(p) \cap \mathrm{supp}(q) = \emptyset$. For $\mathcal{F} \subseteq \mathrm{FIN}_k$ and $p \in \mathrm{FIN}_k$ we also define $\mathcal{F}/p := \{q \in \mathcal{F} : p < q\}$

We say that a family $B \subseteq \mathrm{FIN}_k$ is a *block sequence* if it can be enumerated by natural numbers increasingly with respect to the ordering $<$. Note that there exists exactly one such enumeration, therefore we always assume that any block sequence B comes with its increasing enumeration (b_n), where, unless otherwise stated, we use the convention that elements of a block sequence C are denoted by small letters c equipped with indexes. Last but not least, in the paper, by block sequences we always mean infinite block sequences unless we state otherwise.

For $k \in \mathbb{N}$ define the following *tetris* operation $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ by $T(p)(n) = \max\{p(n)-1, 0\}$, $n < \omega$, and by T^n denote its n^{th} iteration, $n \leqslant k$; of course $T^0 = \mathrm{id}$. The tetris operation is a surjective and additive function which means that $T(p+q) = T(p)+T(q)$ for $p, q \in \mathrm{FIN}_k$. Note that TB (the image of B under T) is a block sequence for any block sequence $B \subseteq \mathrm{FIN}_k$ and $k > 1$.

Every block sequence B generates a set called *combinatorial space* $\langle B \rangle$ defined as the smallest subfamily of FIN_k including B and closed on the summing and tetris operations:

$$\langle B \rangle := \left\{ \sum_{n<\omega} T^{k-f(n)}(b_n) : f \in \mathrm{FIN}_k \right\}.$$

In the above we say that $f \in \mathrm{FIN}_k$ *codes* $\sum_n T^{k-f(n)}(b_n)$ into B. Note that a combinatorial space is a well-defined subfamily of FIN_k and FIN_k is itself such a space generated by *the standard block sequence* $E_k := \{e_n : n < \omega\}$, where $e_n := \{(i, k\delta_{in}) : i < \omega\}$ and δ denotes Kronecker delta. The notion of combinatorial space allows us to consider the following partial ordering on the collection of all block sequences of FIN_k

$$B_0 \preccurlyeq B_1 \quad \text{iff} \quad B_0 \subseteq \langle B_1 \rangle.$$

In this case we say that B_0 is a *block subsequence* of B_1 and it generates a *block subspace* of $\langle B_1 \rangle$. The mapping $\Theta_B : \mathrm{FIN}_k \to \langle B \rangle$ given by $\Theta_B(f) = \sum_n T^{k-f(n)}(b_n)$, $f \in \mathrm{FIN}_k$, defines a combinatorial isomorphism between FIN_k and $\langle B \rangle$ for an arbitrary block sequence $B \subseteq \mathrm{FIN}_k$. More precisely, Θ_B establishes an isomorphism in model-theoretic sense (cf. [6]) between structures $(\mathrm{FIN}_k, +, <, T)$ and $(\langle B \rangle, +, <, T)$. Thus any combinatorial statement about FIN_k remains true for all its block subspaces. Observe also that for any block sequence $B \subseteq \mathrm{FIN}_k$ it holds $\langle TB \rangle = T\langle B \rangle$ which follows from the identity $T(T^i f + T^j g) = T^i(Tf) + T^j(Tg)$ for all $f, g \in \langle B \rangle$ and $(i, j) \in \mathit{Comb}_k$. For $B \subseteq \mathrm{FIN}_k$ as above we define also

$$\langle B \rangle^{(2)} := \left\{ (f, g) \in \langle B \rangle^2 : \ f < g \right\}.$$

More generally $\langle B \rangle^{(d)}$, $d \in \mathbb{N}$, will stand for a collection of finite block subsequences of B of the length d. For a block sequence $B \subseteq \mathrm{FIN}_k$ and a function $p \in \langle B \rangle$ let us define a *relative support* of p with respect to B as $\mathrm{supp}_B(p) := \{b_n \in B : n \in \mathrm{supp}(f)\}$, where $f \in \mathrm{FIN}_k$ codes p into B. For $k > 1$ and a block sequence $B \subseteq \mathrm{FIN}_k$ consider an operation $T_B : \langle TB \rangle \to \langle B \rangle$ given by the following formula

$$T_B \left(\sum_{n<\omega} T^{k-f(n)}(Tb_n) \right) := \sum_{n<\omega} T^{k-f(n)}(b_n), \quad f \in \mathrm{FIN}_{k-1}.$$

Lemma 1 The operation $T_B : \langle TB\rangle \to \langle B\rangle$ satisfies the following

1. T_B is a right inverse of T on $\langle TB\rangle$, i.e. $TT_B = \mathrm{id}_{\langle TB\rangle}$;
2. T_B *respects* B*-supports*: $T\left[\mathrm{supp}_B(T_B p)\right] = \mathrm{supp}_{TB}(p)$ for all $p \in \langle TB\rangle$;
3. T_B is uniquely characterized by 1. and 2.;
4. if $C \preccurlyeq TB$ then $T_B C$ is a block subsequence of B;
5. $T_{B_1} = T_{B_0} \restriction \langle TB_1\rangle$ for $B_1 \preccurlyeq B_0 \preccurlyeq B$;
6. $\langle T_B C\rangle \subseteq T_B\langle C\rangle$ for $C \preccurlyeq TB$.

Proof: We omit the proofs of 1.-5. as they follow almost immediately from the definition of T_B. Only the last point requires some effort. Let $C \preccurlyeq TB$. Evidently $T_B C \subseteq T_B\langle C\rangle$. Thus it suffices to show that for any $p, q \in T_B\langle C\rangle$ and $(i,j) \in Comb_k$ it holds $T^i p + T^j q \in T_B\langle C\rangle$. This however is an immediate consequence of the following two remarks. First, $T_B(p+q) = T_B p + T_B q$ for $p, q \in \langle TB\rangle$, where, of course, the latter sum is well-defined by 2. Second, for $i \in k \setminus 1$ and $p \in \langle C\rangle$ it holds $T^i T_B p \in T_B T^i\langle C\rangle = T_B\langle T^i C\rangle$. The latter is satisfied since if $f \in \mathrm{FIN}_{k-1}$ codes p into C then $T^{i-1} f \in \mathrm{FIN}_{k-i}$ and

$$
\begin{aligned}
T^i T_B p = T^i T_B \sum_n T^{k-1-f(n)}(c_n) &= T^{i-1}(TT_B)\sum_n T^{k-1-f(n)}(c_n) = \\
= \sum_n T^{k-1-(f(n)-(i-1))}(c_n) &= \sum_n T^{k-1-T^{i-1}f(n)}(c_n) = \\
= T_B \sum_n T^{k-1-T^{i-1}f(n)}(Tc_n) &= T_B \sum_n T^{k-T^{i-1}f(n)}(c_n) \in T_B T^i\langle C\rangle. \quad \square
\end{aligned}
$$

A family $\mathcal{F} \subseteq \mathrm{FIN}_k$ is B-*dense* if it meets all block subspaces of $\langle B\rangle$, i.e.

$$\mathcal{F} \cap \langle C\rangle \neq \emptyset \ \text{ for all } \ C \preccurlyeq B.$$

Similarly we define the notion of $B^{(d)}$-denseness. For $d \in \mathbb{N}$ we say that a family $\mathcal{F} \subseteq \mathrm{FIN}_k^{(d)}$ is $B^{(d)}$-dense if $\mathcal{F} \cap \langle C\rangle^{(d)} \neq \emptyset$ for all $C \preccurlyeq B$. Note that for $d = 1$ the latter notion of denseness is the same as the former. In the paper we shall work only with $B^{(d)}$-dense families for $d \in \{1, 2\}$. Futhermore $B^{(d)}$-denseness is a $\preccurlyeq$-hereditary property, that is $B^{(d)}$-dense families are $C^{(d)}$-dense for any $C \preccurlyeq B$. Other basic facts concerning the above definitions are stated in the next lemma (cf [2],[3] while for the notion of a non-principal coideal we refer to [10]).

Lemma 2 For $d \in \mathbb{N}$ and a block sequence $B \subseteq \mathrm{FIN}_k$ the following family forms a non-principal coideal on $\langle B \rangle^{(d)}$

$$LocDense := \left\{ \mathcal{F} \subseteq \langle B \rangle^{(d)} : \mathcal{F} \text{ is } C^{(d)}\text{-dense for some } C \preccurlyeq B \right\}.$$

In other words, for the family *LocDense* it consecutively holds

1. if $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{F} \in LocDense$ then $\mathcal{F}' \in LocDense$;
2. if $|\mathcal{F} \triangle \mathcal{F}'| < \omega$ and $\mathcal{F} \in LocDense$ then $\mathcal{F}' \in LocDense$;
3. if $\mathcal{F} \in LocDense$ is partitioned into finitely many pieces then one of them is in *LocDense*.

Proof: For the sake of clarity in notation we verify $1.-3.$ for the case $d = 1$.

1. trivial;
2. by 1. we can assume that $\mathcal{F}/p \subseteq \mathcal{F}'$ for some $p \in \mathrm{FIN}_k$ and again by 1. it suffices to check that $\mathcal{F}/p$ is B/p-dense; indeed, if it is not the case then $\langle C \rangle \cap \mathcal{F}/p = \emptyset$ for some block sequence $C \preccurlyeq B/p$; then however $\langle C \rangle \cap \mathcal{F} = \langle C \rangle \cap \mathcal{F}/p = \emptyset$ and $C \preccurlyeq B$ witnesses that $\mathcal{F}$ is not B-dense;
3. by simple induction we can restrict ourselves to two-element partitions; if $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ is C-dense while $\mathcal{F}_0$ is not, then $\mathcal{F}_0 \cap \langle C' \rangle = \emptyset$ for some $C' \preccurlyeq C$; hence by C-denseness of $\mathcal{F}$ for any $C'' \preccurlyeq C' \preccurlyeq C$ it holds $\emptyset \neq \mathcal{F} \cap \langle C'' \rangle = \mathcal{F}_1 \cap \langle C'' \rangle$, i.e. $\mathcal{F}_1 \in LocDense$. $\square$

2. FIN_k Theorem and its proof.

The proof of **$\mathbf{FIN}_k$ Theorem**, which we shall give, is by induction on k. Therefore we need write theorems and facts available at an inductive step. To be precise, for $k, d \in \mathbb{N}$ and a block sequence $B \subseteq \mathrm{FIN}_k$ we introduce the following statements $GMTT(k, d, B)$ and $GMTT^*(k, d, B)$.

$GMTT(k,d,B)$: For any $r \in \mathbb{N}$ and for any coloring $c : \langle B \rangle^{(d)} \mapsto r$ there exist $C \preccurlyeq B$ and $i < r$ such that $c \restriction \langle C \rangle^{(d)} \equiv j$.

$GMTT^*(k,d,B)$: For any $B^{(d)}$-dense family $\mathcal{F} \subseteq \mathrm{FIN}_k^{(d)}$ there exists $C \preccurlyeq B$ such that $\langle C \rangle^{(d)} \subseteq \mathcal{F}$.

Let us describe briefly a history. In [5] N.Hindman proved $GMTT(1, 1, E_1)$ ($GMTT^*(1, 1, E_1)$ in fact) which is now known as Hindman Theorem. Soon after, in [2] (cf. [3]), J.E.Baumgartner gave a simpler and more elegant proof of $GMTT^*(1, 1, E_1)$. Both proofs were elementary and did not require

uncountable versions of AC. Multidimensional version of Hindman Theorem, that is $\forall_d GMTT(1, d, E_1)$, was proved in [7] and [9] independently by K.Milliken and A.D.Taylor. This result known as Milliken-Taylor Theorem uses Hindman Theorem as a pigeonhole principle in order to perform a standard diagonalization procedure (eo ipso only a weak countable version of AC is used there). As mentioned in introduction Gowers' **FIN$_k$ Theorem**, that is $\forall_k GMTT(k, 1, E_k)$, was established in [4] with the help of theory of ultrafilters. S.Todorčević pointed out (cf. [10]) that Milliken's and Taylor's proof gives $\forall_{k,d} GMTT(k, d, E_k)$ if the usage of Hindman Theorem as pigeonhole principle is replaced by **FIN$_k$ Theorem**. Therefore the latter theorem we call **Gowers-Milliken-Taylor-Todorčević Theorem**, $GMTT$ Theorem in short. We shall prove it below without referring to any uncountable versions of AC or theirs consequences such as the existence of ultrafilters. In the next lemma we list some relations between $GMTT$ statements.

Lemma 3 For $k, d \in \mathbb{N}$ and a block sequence $B \subseteq \mathrm{FIN}_k$ it holds

1. $GMTT(k, d, E_1) \Leftrightarrow GMTT(k, d, B)$;
2. $GMTT(k, 1, B) \Leftrightarrow GMTT(k, d, B)$;
3. $GMTT(k, d, B) \Leftrightarrow GMTT^*(k, d, B)$.

Proof:

1. follows from the fact that all combinatorial spaces are isomorphic;
2. follows from a diagonalization procedure mentioned above;
3. assume $GMTT(k, d, B)$ and let $\mathcal{F} \subseteq \mathrm{FIN}_k^{(d)}$ be $B^{(d)}$-dense family; consider a coloring $c : \langle B\rangle^{(d)} \mapsto 2$ given by $c(\overline{p}) = 0$ iff $\overline{p} \in \mathcal{F}$, $\overline{p} \in \langle B\rangle^{(d)}$; take $C \preccurlyeq B$ such that $c \restriction \langle C\rangle^{(d)} \equiv i$ for some $i < 2$; note that $i = 1$ implies $\langle C\rangle^{(d)} \cap \mathcal{F} = \emptyset$ contradicting $B^{(d)}$-denseness of $\mathcal{F}$; hence $i = 0$ which means that $\langle C\rangle^{(d)} \subseteq \mathcal{F}$ and $GMTT^*(k, d, B)$ follows from the arbitrariness of $\mathcal{F}$;
   assume $GMTT^*(k, d, B)$ and let $c : \langle B\rangle^{(d)} \mapsto r$ be an arbitrary coloring, $r \in \mathbb{N}$; since $B^{(d)}$-dense family $\langle B\rangle^{(d)}$ can be partitioned into the sets $c^{-1}[\{i\}]$, $i < r$, **Lemma 2.3** guarantees the existence of $i < r$ such that $c^{-1}[\{i\}]$ is $B'^{(d)}$-dense for some $B' \preccurlyeq B$; thus there is $C \preccurlyeq B'$ with $\langle C\rangle^{(d)} \subseteq c^{-1}[\{i\}]$ which means that $c \restriction \langle C\rangle^{(d)} \equiv i$ and so $GMTT(k, d, B)$ follows from the arbitrariness of c. $\square$

Observe that the above proof and the ones below also use no uncountable form of AC. The previous and forthcoming lemmas culminate in a crucial **Lemma 6** which will be the main tool in our proof of **FIN$_k$ Theorem**.

Lemma 4 Let $B \subseteq \mathrm{FIN}_k$ be a block sequence, $k \in \mathbb{N}$. If a family $\mathcal{F} \subseteq \mathrm{FIN}_k$ is B-dense then for all $(i,j) \in \mathit{Comb}_k$ and for any $C \preccurlyeq B$ one can find $(p,q) \in \langle C \rangle^{(2)}$ such that $T^i p + T^j q \in \mathcal{F}$.

Proof: Fix $k \in \mathbb{N}$, a block sequence $B \subseteq \mathrm{FIN}_k$, B-dense family $\mathcal{F} \subseteq \mathrm{FIN}_k$, $(i,j) \in \mathit{Comb}_k$ and a block sequence $C \preccurlyeq B$. Consider a case $j = 0$. Take a block subsequence $C' \preccurlyeq C$ defined as follows

$$c'_n := \sum_{l=0}^{i} T^{i-l} c_{(i+2)n+l} + c_{(i+2)(n+1)-1}, \quad n < \omega.$$

As $C' \preccurlyeq B$ and the family $\mathcal{F}$ is B-dense we can find $r \in \mathcal{F} \cap \langle C' \rangle$. Let $f \in \mathrm{FIN}_k$ be the code of r in C' and put the following numbers

$$n_* := \min \operatorname{supp}(f) \text{ and } l_* := \min\{l \leqslant i : k - f(n_*) + i - l < k\}.$$

It is a straightforward yet cumbersome verification that the functions below

- $p := \sum_{l=l_*}^{k-f(n_*)} T^{k-f(n_*)-l} c_{(i+2)n_*+l}$,
- $q := \sum_{l=k-f(n_*)+1}^{i} T^{k-f(n_*)+i-l} c_{(i+2)n_*+l} + \sum_{n>n_*} T^{k-f(n)} c'_n$,

satisfy $p, q \in \langle C \rangle$, $p < q$ and $r = T^i p + q \in \mathcal{F}$. The other case $i = 0$ can be handled similarly by a mirror reflection of definitions of C', n_*, l_*, p, q. $\square$

Lemma 5 Let $B \subseteq \mathrm{FIN}_k$ be a block sequence, $k \in \mathbb{N}$. If a family $\mathcal{F} \subseteq \mathrm{FIN}_k$ is B-dense then for all $(i,j) \in \mathit{Comb}_k$ the following family is $(T^{i+j}B)^{(2)}$-dense.

$$\mathcal{F}(i,j) := \left\{(p,q) \in \langle T^{i+j} B \rangle^{(2)} : T^i_B p + T^j_B q \in \mathcal{F}\right\}$$

Proof: Fix $k \in \mathbb{N}$, a block sequence $B \subseteq \mathrm{FIN}_k$, B-dense family $\mathcal{F} \subseteq \mathrm{FIN}_k$, $(i,j) \in \mathit{Comb}_k$ and an arbitrary block sequence $C \preccurlyeq T^{i+j}B$. By a repeated application ($i+j$ times) of **Lemma 1.4** we infer that a block sequence $T^{i+j}_B C$ is a block subsequence of B. Using **Lemma 4** we can find $(p', q') \in \langle T^{i+j}_B C \rangle^{(2)}$ with $T^j p' + T^i q' \in \mathcal{F}$. Next, by a repeated application (again $i+j$ times) of **Lemma 1.6** we obtain $\langle T^{i+j}_B C \rangle \subseteq T^{i+j}_B \langle C \rangle$. Hence there exists a pair $(p,q) \in \langle C \rangle^{(2)}$ with $(T^{i+j}_B p, T^{i+j}_B q) = (p', q')$. Finally, by **Lemma 1.1** we get

$$T^i_B p + T^j_B q = T^j T^{i+j}_B p + T^i T^{i+j}_B q = T^j p' + T^i q' \in \mathcal{F}.$$

As $C \preccurlyeq T^{i+j}B$ was arbitrary, the result follows. $\square$

In order to state the next result in a more suggestive way let us make the following ad hoc definition. For $k \in \mathbb{N}$, a block sequence $B \subseteq \mathrm{FIN}_k$ and a pair $(i,j) \in \mathit{Comb}_k$ we say that a family $\mathcal{F} \subseteq \mathrm{FIN}_k$ is (i,j)-*closed over*

B if $T^i p + T^j q \in \mathcal{F}$ for any $(p,q) \in \langle B\rangle^{(2)}$. Observe that this notion is $\preccurlyeq$-hereditary, i.e. if $\mathcal{F} \subseteq \mathrm{FIN}_k$ is (i,j)-closed over B and $C \preccurlyeq B$ then $\mathcal{F}$ is (i,j)-closed over C as well.

Lemma 6 Let $B \subseteq \mathrm{FIN}_k$ be a block sequence, $k \in \mathbb{N}$. If a family $\mathcal{F} \subseteq \mathrm{FIN}_k$ is B-dense and $G^*(i+j,2)$ holds for $(i,j) \in Comb_k$ then there exists a block subsequence $C \preccurlyeq B$ such that $\mathcal{F}$ is (i,j)-closed over C.

Proof: Fix $k \in \mathbb{N}$, a block sequence $B \subseteq \mathrm{FIN}_k$, B-dense family $\mathcal{F} \subseteq \mathrm{FIN}_k$ and $(i,j) \in Comb_k$. By **Lemma 5** a family $\mathcal{F}(j,i)$ is $(T^{i+j}B)^{(2)}$-dense. Using $G^*(i+j,2)$ we can conclude the existence of a block subsequence $B' \preccurlyeq T^{i+j}B$ such that $\langle B'\rangle^{(2)} \subseteq \mathcal{F}(j,i)$. We claim that $C := T_B^{i+j}B'$ (which is a block sequence by an $(i+j)^{\text{th}}$ iteration of **Lemma 1.4**) is the sought block sequence. Indeed, let $(p,q) \in \langle C\rangle^{(2)}$ be arbitrary. Using $\langle T_B^{i+j}B'\rangle \subseteq T_B^{i+j}\langle B'\rangle$ as in **Lemma 5**, we can find pair $(p',q') \in \langle B'\rangle^{(2)}$ such that $(T_B^{i+j}p', T_B^{i+j}q') = (p,q)$. By the choice of B' we obtain $(p',q') \in \mathcal{F}(j,i)$, i.e. $T_B^j p' + T_B^i q' \in \mathcal{F}$. Thus

$$T^i p + T^j q = T^i T_B^{i+j} p' + T^j T_B^{i+j} q' = T_B^j p' + T_B^i q' \in \mathcal{F}.$$

It ends the proof of the lemma as $(p,q) \in \langle C\rangle^{(2)}$ was arbitrary. $\square$

FIN_k Theorem For every finite coloring of FIN_k there is a block sequence $B \subseteq \mathrm{FIN}_k$ such that a combinatorial space $\langle B\rangle$ is monochromatic.

Proof of FIN_k Theorem: The above theorem is the statement $G(k,1,E_k)$. We shall prove by induction on k that for all $k \in \mathbb{N}$ it holds $G^*(k,1,E_k)$, which is equivalent to $G(k,1,E_k)$ by **Lemma 2**. The base case $k=1$ is provided by Hindman Theorem with its original or Baumgartner's proof. Let $k \in \mathbb{N}$ and assume that $G^*(l,1,E_l)$ holds for $l \in k \setminus 1$. Observe that again by **Lemma 2** for every $l \in k \setminus 1$ as well as for every block sequences $B \subseteq \mathrm{FIN}_l$ the statement $G^*(l,2,B)$ is also provided by the induction hypothesis.

Let $\mathcal{F} \subseteq \mathrm{FIN}_k$ be an arbitrary E_k-dense family. Enumerate $Comb_k$ into $\{(i_m,j_m) : 0 < m < 2k\}$. Starting with $B_0 := E_k$ we build inductively a $\preccurlyeq$-decreasing sequence $(B_m : m < 2k)$ of block subsequences of E_k such that for all $0 < m < 2k$ the family $\mathcal{F}$ is (i_m,j_m)-closed over B_m. This is done by induction on m as follows. Suppose that we have already constructed a partial sequence $(B_l : l < m)$, $m < 2k-1$, with the above properties. As $B_{m-1} \preccurlyeq B$ and $\mathcal{F} \subseteq \mathrm{FIN}_k$ is B-dense it is also B_{m-1}-dense. Since $i_{m-1} + j_{m-1} < k$ the statement $G^*(i_{m-1}+j_{m-1}, 2, B_{m-1})$ holds by the induction hypothesis. Therefore the existence of the needed block subsequence $B_m \preccurlyeq B_{m-1}$ is guaranteed by **Lemma 6**. This finishes the construction of $(B_m : m < 2k)$. Finally, define $b_n = b'_{2n} + b'_{2n+1}$, $n < \omega$, where $B_{2k-1} = (b'_n : n < \omega)$ (in fact any block subsequence of B_{2k-1} and disjoint from it would suffice). Recall our

convention that (b_n) and B denote the same block sequence (unless stated otherwise).

We claim that B satisfies $\langle B\rangle \subseteq \mathcal{F}$. Firstly, note that $B \subseteq \mathcal{F}$ since $\mathcal{F}$ is $(0,0)$-closed over B_{2k-1}. Secondly, the family $\mathcal{F}$ is (i,j)-closed over B for every $(i,j) \in Comb_k$. This a consequence of definition of B_m's, a $\preccurlyeq$-hereditary character of the notion of closedness and the fact that $B \preccurlyeq B_m$ for all $m < 2k$. Having these two properties of B in mind, we shall show that any $p \in \langle B\rangle$ is also in $\mathcal{F}$ by induction on cardinality of $\mathrm{supp}_B(p)$ or equivalently by induction on cardinality of a support of the code of p in B. The base case $n = 1$ is provided by the inclusion $B \subseteq \mathcal{F}$. Let $n \in \mathbb{N}$ and suppose that any function from $\langle B\rangle$ is also in $\mathcal{F}$ if it is coded in B by a function $f \in \mathrm{FIN}_k$ with $|\mathrm{supp}(f)| \leqslant n$. Let us name the latter assumption by an *outside induction hypothesis.* Let $f \in \mathrm{FIN}_k$ be the code in B of $p \in \langle B\rangle$ such that $|\mathrm{supp}(f)| = n + 1$ and choose $n_0 \in \mathrm{supp}(f)$ with $f(n_0) = k$. Then one of the sets $\mathrm{supp}(f) \cap n_0$, $\mathrm{supp}(f) \setminus (n_0 + 1)$ is nonempty. Without loss of generality assume that $\mathrm{supp}(f) \cap n_0 \neq \emptyset$. Enumerate $<$-decreasingly $\mathrm{supp}(f) \cap (n_0 + 1)$ into $\{n_m : m < N\}$ where $N := |\mathrm{supp}(f) \cap (n_0 + 1)| \geqslant 2$. For $m < N$ define a function $f_m \in \mathrm{FIN}_k$ given by the following conditions

$$\mathrm{supp}(f_m) = \mathrm{supp}(f) \setminus n_m \text{ and } f_m \restriction \mathrm{supp}(f_m) = f \restriction \mathrm{supp}(f_m).$$

For $m < N$ denote by $p_m \in \langle B\rangle$ a function coded by f_m into B. We now prove by induction on m that $\{p_m : m < N\} \subseteq \mathcal{F}$. The base case $m = 0$ follows from $|\mathrm{supp}(f_0)| \leqslant n$ (as $\mathrm{supp}(f) \cap n_0 \neq \emptyset$) and from the outside induction hypothesis. Let $m < N - 1$ and suppose that $p_m \in \mathcal{F}$. This will be called an *inside induction hypothesis.* Observe that $p_{m+1} = T^{k-f(n_{m+1})} b_{n_{m+1}} + p_m$. From $b_{n_{m+1}} \in B \subseteq \mathcal{F}$, $p_m \in \mathcal{F}$ (by the inside induction hypothesis) and from the fact that $\mathcal{F}$ is $(k - f(n_{m+1}), 0)$-closed over B, we obtain $p_{m+1} \in \mathcal{F}$. This obviously ends the inside inductive proof as well as the outside inductive proof as $p_{N-1} \in \mathcal{F}$ is coded into B by the function $f_{N-1} = f$. Hence the inclusion $\langle B\rangle \subseteq \mathcal{F}$ is established as well as our main goal $G^*(k, 1, E_k)$. $\square$

GMTT Theorem For every $k, d \in \mathbb{N}$ and every finite coloring of $\mathrm{FIN}_k^{(d)}$ there is a block sequence $B \subseteq \mathrm{FIN}_k$ such that $\langle B\rangle^{(d)}$ is monochromatic.

Proof: It follows from **FIN$_k$ Theorem** and **Lemma 3**. $\square$

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